The *k*-fold list coloring of cycles with Hall's condition

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Abstract. We prove that any cycle C_n, $n \geq 4$, with list assignment *L*, has a *k*-fold list coloring from the given lists if (*i*) each list contains at least 2*k* colors and (*ii*) C_n and *L* satisfy Hall's condition for *k*-fold list colorings. Further, 2*k* in (*i*) cannot be replaced by $2k-1$ if either *n* is odd, or *n* is even and $n \geq k+2$. In other words, if $n \geq 4$, the *k*-fold Hall number of a cycle C_n satisfies $h^{(k)}(C_n) \leq 2k$, with equality if *n* is odd, or *n* is even and $n \geq k+2$.

Key words and phrases: Hall's condition, Hall number, list coloring, list assignment, *k*-fold choice number, *k*-fold chromatic number.

1. Introduction

A *list assignment*, or a *color supply*, for a graph *G* = (*V*, *E*) is an assignment to the vertices of *G* of finite subsets ("lists") of a set *C* of colors. A *color demand* for *G* is an assignment of positive integers to the vertices of *G*. If *L* is a color supply and *w* is a color demand for *G*, an (*L*,*w*)-coloring of *G* is a function φ which assigns to each $v \in V$ a subset $\varphi(v) \subseteq L(v)$, with $|\varphi(v)| = w(v)$, such that $\varphi(u) \cap \varphi(v) = \varnothing$ whenever *uv* is an edge in *G*. When *w* is a constant function, say *w* = *k*, we let the value *k* stand for the function *w* and we may refer to an (*L*,*k*)-coloring of *G* as a *k*-fold list coloring from *L*.

For any positive integer *k* the *k-fold chromatic number* of *G*, denoted $\chi^{(k)}(G)$, is the smallest integer *m* such that there is a *k*-fold list coloring from the constant color supply $L \equiv \{1,...,m\}$. The *k-fold choice number* of *G*, denoted $ch^{(k)}(G)$, is the smallest integer *m* such that there is a *k*-fold list coloring from any color supply *L* satisfying $|L(v)| \geq m$ for all $v \in V$.

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The idea of coloring the vertices of a graph with subsets of a fixed set arguably originated with Hilton, Rado, Scott, or Stahl ([14], [17], [19]). The k-fold coloring of cycles is a particular case of edge coloring of multicycles which was studied by Kostochka and Woodall in [15]. It turns out that for cycles the k-fold chromatic and choice numbers are the same: Tuza and Voigt in [20] and Gutner and Tarsi in [6] proved that $ch^{(k)}(C_{2m}) = 2k$, $m = 1, 2, \ldots$, $k = 1, 2, \ldots$, and Slivnik [18] used measure theoretic methods to show that $ch^{(k)}(C_{2m+1}) = 2k + \lceil k/m \rceil$; these values had long been known for $\chi^{(k)}(G)$. Our aim here is to give an upper bound for cycles, with equality in many cases, on another k-fold list coloring parameter whose definition involves a fairly well-known necessary condition for the existence of an (L,w)-coloring of G called Hall's condition. Our main result will imply the results above about $ch^{(k)}$ as a corollary.

Given G, L, w, and induced subgraph H of G, and a color $x \in C$, let $H(x,L)$ denote the subgraph of H induced by $\{v \in V(H) | x \in L(v)\}$, and let $\alpha(H(x,L))$ denote the vertex independence number of this subgraph. [If x does not appear on any lists $L(v)$, $v \in V(H)$, set $\alpha(H(x,L)) = 0$.] If there is an (L,w)-coloring φ of G then the set of vertices $v \in V(H)$ such that $x \in \varphi(v)$ is an independent set of vertices in $H(x,L)$. Therefore, there can be no more than $\alpha(H(x,L))$ appearances of x in the color sets for vertices of H. Since the total number of appearances of all symbols in those color sets is $\sum_{x \in V(H)} |\varphi(v)| = \sum_{v \in V(H)} w(v)$, we have

$$(*) \qquad \sum_{x \in C} \alpha(H(x,L)) \geq \sum_{v \in V(H)} w(v).$$

If (*) holds for every induced subgraph H of G, then G, L, and w satisfy *Hall's condition*. (NOTE: if G, L, and w satisfy Hall's condition, then (*) holds for every subgraph H of G, induced or not.) Observe that when $w = k$, a constant, the right hand side of (*) is $k|V(H)|$.

Hall's condition is so named because it was inspired by Hall's theorem on systems of distinct representatives [7], which can be viewed as a list coloring theorem about the special case when G is a clique and $w = 1$ [10]. Further, the improvement of Hall's theorem, in which the requirement that $w = 1$ is removed, due to Rado [16] and, independently, to Halmos and Vaughan [8], can be stated thus: when G is a clique, Hall's condition suffices for the existence of an (L,w)-coloring of G.

For each positive integer *k*, the *k*-fold Hall number of *G*, denoted $h^{(k)}(G)$, is the smallest integer $m \geq k$ such that there is an (*L*,*w*)-coloring of *G* whenever *G*, *L*, and *k* satisfy Hall's condition and $|L(v)| \geq m$ for all $v \in V$. Note that $h^{(k)}(G) \geq k$ for every *G* and *k* and the Hall – Rado – Halmos – Vaughan theorem referred to above implies equality, for every *k*, when *G* is a clique. It follows from results in [2], [5] and [13] that: $h^{(k)}(G) = k$ for $k = 1, 2, \ldots$ if and only if *G* is the line graph of a forest.

Clearly $h^{(k)}(G) \leq ch^{(k)}(G)$ for all *k* and *G*, but $h^{(k)}(G) < \chi^{(k)}(G)$, $h^{(k)}(G) = \chi^{(k)}(G)$, and $h^{(k)}(G) > \chi^{(k)}(G)$ are all possible [5]. In [5] it is shown that $h^{(k)}(G) = ch^{(k)}(G)$ if either is larger than $\chi^{(k)}(G)$. This is an easy result, but it may have its uses, since both $ch^{(k)}(G)$ and $h^{(k)}(G)$ are difficult to determine.

For $k \geq 2$, the problem of determining the *k*-fold Hall numbers of trees is solved in [3]: if $r \geq 3$, $h^{(k)}(K_{1,r}) = 2k - \lfloor k/r \rfloor$; if $r < 3$, $h^{(k)}(K_{1,r}) = k$; and if *T* is a tree which is not a star, $h^{(k)}(T) = 2k$. It follows from the main result in [10] that $h^{(1)}(T) = 1$ for all trees *T*.

Regarding cycles, since $C_3 = K_3$ we have $h^{(k)}(C_3) = k$ for all $k = 1, 2 \ldots$. In [3], we proved that $h^{(k)}(C_4) = \left\lceil \frac{5k}{3} \right\rceil$ for all $k = 1, 2, \ldots$. From [13] we have $h^{(1)}(C_n) = 2$ for all $n > 3$. Our main result is the following:

Theorem. For any integers $n \geq 4$ and $k \geq 1$, $h^{(k)}(C_n) \leq 2k$, with equality if *n* is odd, or if *n* is even and $n \geq k + 2$.

For *n* even the inequality $h^{(k)}(C_n) \leq 2k$ follows from $h^{(k)} \leq ch^{(k)}$ and the previously cited result from [6] and [20] that $ch^{(k)}(C_n) = 2k$. But our proof will be independent of all direct proofs of this fact, and that allows us to turn the tables and give a new proof of both old results about $ch^{(k)}(C_n)$, *n* even or odd. The following is a straightforward consequence of the previously mentioned result in [5] that $ch^{(k)} = \max[\chi^{(k)}, h^{(k)}]$, of the well known values of $\chi^{(k)}(C_n)$, $k \geq 1$, $n \geq 3$, and of the theorem.

Corollary. For any integers $k \geq 1$, $m \geq 2$, $ch^{(k)}(C_{2m}) = 2k$ and $ch^{(k)}(C_{2m+1}) = 2k + \lceil k/m \rceil$.

In the case of C_{2m+1}, the proof of the corollary constitutes a purely combinatorial proof of the result proved by Slivnik [18] with the involvement of Lebesgue measure.

There is an open problem in [5] that our results bear on : does $\lim_{k\to\infty} \frac{h^{(k)}(G)}{k}$ exist for every graph G? To the two classes of graphs for which the answer was known to be yes – line graphs of forests and trees – we can now add: odd cycles. It feels strange to wrestle with a list coloring problem that is apparently harder for even cycles than for odd cycles! We hope to throw some further light on $h^{(k)}(C_{2m})$ when $2m \le k+1$ in a forthcoming paper.

2. Proof of the Theorem

The proof that $h^{(k)}(C_n) \le 2k$ will use the following, a special case of the main result in [2], first proved in [1].

Path Lemma. For any path *P* with color demand *w* and color supply *L*, *P* is (*L*,*w*)-colorable if and only if *P*, *L*, and *w* satisfy Hall's condition.

It may be worth mentioning that in [1] there is an efficient algorithm for either coloring *P* or discovering that Hall's condition is not satisfied.

Our strategy for proving that $h^{(k)}(C_n) \le 2k$ will use the Path Lemma thus: we will show that when C_n, *L* and *k* satisfy Hall's condition and $|L(v)| \ge 2k$ for all $v \in V(C_n)$, then for some $v \in V(C_n)$ there is a *k*-set $S \subseteq L(v)$ such that if L' is defined on the path $C_n - v$ by removing all of the elements of *S* from the lists on the neighbors of *v*, and otherwise putting $L' = L$, then $C_n - v$, L' and *k* satisfy Hall's condition. Coloring *v* with *S* and putting this with an (L', *k*)-coloring of $C_n - v$ then produces an (*L*,*k*)-coloring of C_n.

It will be useful to note as in [1] that for any list assignment *L* to C_n and any $x \in C$, we may as well suppose that $C_n(x, L)$ is connected. The reason: for any graph *G*, color supply *L*, and color demand *w*, if *G*(*x*,*L*) is disconnected for some *x* in *C*, we can make a new list assignment $\hat{L}$ by replacing *x* on the lists on each component of *G*(*x*,*L*) by a new symbol, so that *x*'s replacements on those components are different from each other and from all the other symbols appearing on lists on *G*. It is straightforward to see that *G*, *L*, and *w* satisfy Hall's condition if an only if G, $\hat{L}$, and w do, and that there is an (L,w)-coloring of G if and only if there is an ($\hat{L}$ w)-coloring of G.

Given a vertex v of $G = C_n$ and a color $x \in L(v)$, we say that x is *bad at v* if and only if no maximum independent set of vertices of $G(x,L)$ contains v. Under the assumption that $G(x,L)$ is connected, this means that $G(x,L)$ is a path of odd order and v is one of the *even* numbered vertices, if we count along the path with the count starting with the number 1 (since the path has odd order, you can start from either end).

For each vertex $v \in V(G)$ we partition its supply, $L(v)$, as follows:

$B(v) = \{x \in L(v)\colon x \text{ is bad at } v\}$

$O(v) = \{x \in L(v)\colon G(x,L) \text{ is a path of odd order}\}\backslash B(v)$

$E(v) = L(v)\backslash(B(v) \cup O(v))$.

If $G(x,L)$ is simply a vertex v then $x \in O(v)$. If $G(x,L) = G$ then $x \in E(v)$. We now assume that G, L and k satisfy Hall's condition, and that $|L(v)| \geq 2k$ for all $v \in V(G)$, and set about showing that there is an (L,k)-coloring, by the strategy described earlier.

If $x \in B(v)$ then $x \in O(u)$ for each neighbor u of v. Therefore, if $|O(v) \cup E(v)| < k$ then $|O(u)| \geq |B(v)| > k$ for each neighbor u of v, since $|L(u)| \geq 2k$ for every vertex. It follows that there is a vertex v_0 such that $|O(v_0) \cup E(v_0)| \geq k$. Let the vertices of G be $v_0, v_1, \ldots, v_{n-1}$, one way or the other around the cycle.

Let $X_0 \subseteq O(v_0) \cup E(v_0)$ be of size k and such that $|X_0 \cap O(v_0)|$ is as large as possible. We intend to color v_0 with X_0. Define L' on G-v_0 by

$$L'(v_i) = \begin{cases} L(v_i) \setminus X_0 & \text{if } i = 1, n-1 \\ L(v_i) & \text{otherwise.} \end{cases}$$

We shall finish the proof that $h^{(k)}(C_n) \leq 2k$ by showing that G-v_0, L' and k satisfy Hall's condition.

Let H be an induced subgraph of G-v_0. To verify (*) for H, L' and $w = k$, it suffices to verify it for every connected component of H, so we may as well consider H to be connected; that is, H is a path. If H is a single vertex v_i, $\sum_{x \in C} \alpha(H(x, L')) = |L'(v_i)|$, which is either at least $2k$, if $1 < i < n-1$, or is $|L(v_i) \setminus X_0| \geq 2k - k = k$, if $i \in \{1, n-1\}$. In any case,

$\sum_{x\in C}\alpha(H(x,L')) \ge k = k\,|V(H)|$, so (*) holds and we assume that $\left|V\left(H\right)\right| > 1$. If *H* is a path containing neither v_1 nor v_{n-1} then (*) holds because $L = L'$ on *V*(*H*) and *G*, *L*, and *k* are assumed to satisfy Hall's condition. Therefore, we need only consider the case that *H* is a path containing either v_1 or v_{n-1}, or both.

Suppose that *H* contains v_1 but not v_{n-1}. (Disposing of this case will also take care of the case when *H* contains v_{n-1} but not v_1.) Then, using self-explanatory notation for paths, $H = (v_1, v_2, \ldots, v_t)$, for some *t*, $1 < t < n-1$. Since $v_t, v_{t-2}, \ldots, v_{t-2r}$, where $r = \left\lfloor \frac{t-1}{2} \right\rfloor$, are independent vertices in *H*, we have that

$$\begin{aligned}\sum_{x\in C}\alpha(H(x,L')) &\ge \sum_{i=0}^{r} |L'(v_{t-2i})| \\ &\ge \begin{cases} 2rk+k & \text{if } t \text{ is odd} \\ 2(r+1)k & \text{if } t \text{ is even} \end{cases} \\ &= tk = k|V(H)|.\end{aligned}$$

Now suppose that $H = (v_1, v_2, \ldots, v_{n-1}) = G - v_0$.

If $x \in C \setminus X_0$ then $\alpha(H(x,L')) = \alpha(H(x,L))$. Also, the definition of $O(v_0)$ implies that, for every $x \in O(v_0)$, $\alpha(H(x,L')) = \alpha(H(x,L)) = \alpha(G(x,L)) - 1$. If $|O(v_0)| \ge k$ then X_0 is a subset of $O(v_0)$ so we have that $\alpha(H(x,L')) = \alpha(H(x,L))$ for all *x*, and we are done by the assumption that *G*, *L* and *k* satisfy Hall's condition.

Therefore, we may assume that $|O(v_0)| < k$ and, consequently, by the choice of X_0, that $O(v_0) \subseteq X_0$. As noted above, if $x \in O(v_0)$, $\alpha(H(x,L')) = \alpha(H(x,L)) = \alpha(G(x,L)) - 1$. For $x \in X_0 \setminus O(v_0) \subseteq E(v_0)$, we also have that $\alpha(H(x,L')) \ge \alpha(G(x,L)) - 1$, by the definition of $E(v_0)$ and of L'. Thus $\alpha(H(x,L')) \ge \alpha(G(x,L)) - 1$ for all $x \in X_0$. If $x \in C \setminus X_0$ then either $x \notin L(v_0)$, in which case $\alpha(H(x,L')) = \alpha(G(x,L))$, or $x \in B(v_0) \cup (E(v_0) \setminus X_0)$, in which case $\alpha(H(x,L')) = \alpha(G(x,L))$, as well. Consequently,

$$\begin{aligned}\sum_{x\in C}\alpha(H(x,L')) &= \sum_{x\in X_0}\alpha(H(x,L')) + \sum_{x\in C\setminus X_0}\alpha(H(x,L')) \\ &\ge \sum_{x\in X_0}\alpha(G(x,L)) - |X_0| + \sum_{x\in C\setminus X_0}\alpha(G(x,L))\end{aligned}$$

$$= \sum_{x \in C} \alpha(G(x,L)) - k \ge kn - k = k\,|V(H)|.$$

This completes the proof $h^{(k)}(C_n) \le 2k$.

Next we show that $h^{(k)}(C_{2m+1}) = 2k$ for all $k \ge 1$ and $m \ge 2$ by exhibiting a color supply L satisfying Hall's condition with $G = C_{2m+1}$ and $w = k$, and $|L(v)| \ge 2k-1$ for all *v*, such that there is no (*L*,*k*)-coloring of *G*. Let $v_0, v_1, \ldots, v_{2m}$ be the vertices of *G* around the cycle. For any positive integer *z*, let [*z*]={1, 2, …, *z*} and define $L(v_0) = [2k-1]$, $L(v_1) = L(v_{2m}) = [2k]$, and $L(v_j) = [2k] + (2k-1)$ for $2 \le j \le 2m-1$, where [*z*]+*x* = {1+*x*, 2+*x*, …, *z*+*x*}.

In any (*L*,*k*)-coloring of *G*, because the lists $L(v_j)$, $2 \le j \le 2m-1$, have only 2*k* elements, color 2*k* would have to appear on either v_2 or v_{2m-1}. On the other hand, in every (*L*,*k*)-coloring of (v_1, v_0, v_{2m}) color 2*k* must be used on both v_1, and v_{2m}. Therefore, no such coloring of the cycle exists. It is straightforward to verify that there is an (*L*,*k*)-coloring of *G*-*v* for every $v \in V$; therefore (*) holds for every proper induced subgraph *H* of *G*. Thus we need only to verify (*) for *H* =*G*. The following are easily seen: for $x \in [2k-1]$, $\alpha(G(x,L))$=2; $\alpha(G(2k,L)) = m$ and for $x \in [2k-1] + 2k$, $\alpha(G(x,L)) = m-1$. Therefore,

$$\sum_{x \in C} \alpha(G(x,L)) = 2(2k-1) + m + (m-1)(2k-1) = k(2m+1) + k - 1 \ge k(2m+1) = k|V(G)|.$$ It may be worth pointing out that the list assignment given to show that $h^{(k)}(C_{2m+1}) \ge 2k$ does not do the same job for C_{2m}, $m \ge 2$, for the reason that there is an (*L*,*k*)-coloring of C_{2m}, if *L* is defined as above.

To finish the proof of the Theorem, we suppose that $m \ge \left\lceil \frac{k+2}{2} \right\rceil$, so that $n = 2m > k+1$, and we give a list assignment *L* to $G = C_{2m}$ such that *G*, *L*, and *k* satisfy Hall's condition, and $|L(v)| \ge 2k-1$ for all $v \in V$, yet there is no (*L*,*k*)-coloring of *G*. Let the vertices of *G* be $v_0, v_1, \ldots, v_{2m-1}$, around the cycle.

Case 1. $m \le k$: set $L(v_0) = [2k]$, $L(v_j) = [2k-1] + j$, $1 \le j \le 2m-3$, $L(v_{2m-2}) = [2k] + (2m-3)$ and

$$L(v_{2m-1}) = \{1,3,5,\ldots,2m-3\} \cup \{2k+1, 2k+3, 2k+5, \ldots, 2k+2m-3\} \cup ([k-1] + (2k+2m-3)).$$

Case 2. $m \geq k+1$: set $L(v_0) = [2k]$, $L(v_j) = [2k-1] + j$, $1 \leq j \leq 2k-1$, $L(v_j) = [2k] + (2k-1)$, $2k \leq j \leq 2m-2$, and

$L(v_{2m-1}) = \{1,3,5,...,2k-1\} \cup \{2k+1, 2k+3, 2k+5,...,4k-1\} \cup ([k-1] + (4k-1))$.

First note that all lists have size at least $2k-1$. The only unobvious case is that of $L(v_{2m-1})$ in Case 1, where we have

$|L(v_{2m-1})| = m-1+m-1+k-1 = 2m+k-3 \geq k+2+k-3 = 2k-1$.

Next we shall see that there is no (*L*,*k*)-coloring of *G*. If φ were such a coloring, then, because $|L(v_j) \cup L(v_{j+1})| = 2k$, $0 \leq j \leq 2m-3$, it must be that $\varphi(v_j) = L(v_j) \setminus \varphi(v_{j+1})$, $0 \leq j \leq 2m-3$, and $\varphi(v_j) = L(v_j) \setminus \varphi(v_{j-1})$, $1 \leq j \leq 2m-2$. Related observations: for $0 \leq j \leq 2m-4$ in Case 1 and for $0 \leq j \leq 2k-2$ in Case 2, it must be that $j+1 \in \varphi(v_j)$. Otherwise, $\varphi(v_j) \subseteq L(v_{j+1})$ and we would have $|\varphi(v_{j+1})| = |L(v_{j+1}) \setminus \varphi(v_j)| = 2k-1-k = k-1$. By the same argument, slightly modified, $2m-2 \in \varphi(v_{2m-3})$ in Case 1 and $2k \in \varphi(v_{2k-1})$ in Case 2. Similarly, for $2 \leq j \leq 2m-2$ in Case 1 and for $2 \leq j \leq 2k$ in Case 2, it must be that $2k-1+j \in \varphi(v_j)$.

Using these observations with $\varphi(v_j) = L(v_j) \setminus \varphi(v_{j\pm1})$ for various values of *j*, it can be seen that, in Case 1, $\varphi(v_0)$ must contain 1, 3, …, 2*m*-3, and $\varphi(v_{2m-2})$ must contain $2m+2k-3, 2m+2k-5,...,2k+1$. But this leaves only *k*-1 colors in $L(v_{2m-1})$ eligible for $\varphi(v_{2m-1})$. So there is no such φ in Case 1. In Case 2, something similar happens: $\varphi(v_0)$ must contain 1, 3, …, 2*k*-1, and $\varphi(v_{2m-2})$ must contain $4k-1$, $4k-3$, …, $2k+1$, which leaves only $k-1$ elements in $L(v_{2m-1})$ eligible for $\varphi(v_{2m-1})$.

To show that *G*, *L*, and $w = k$ satisfy Hall's condition is straightforward, but enough of a chore that we shall bear some of the burden here. First we show that *G*-*v* has an (*L*,*k*)-coloring for each $v \in V$. From the discussion of the non-existence of φ, above, it is clear that this holds for $v = v_{2m-1}$, in both cases, but for other $v \in V$ there is work to be done. We start by explicitly defining a "near" *k*-fold coloring of *G* from the list assignment *L*, the very φ whose values on $v_0, v_1, \ldots, v_{2m-2}$ are forced in Case 2 and partially in Case 1. For each $v \in V$, let *odd*(*v*) and *even*(*v*) denote respectively the subsets of odd and even elements in *L*(*v*), and for $0 \leq j < 2m-2$,

$$\varphi(v_j)=\begin{cases} odd(v_j) \text{ if } j \text{ is even} \\ even(v_j) \text{ if } j \text{ is odd}\end{cases}; \text{ further, } \varphi(v_{2m-1})=\begin{cases}[k-1]+(2k+2m-3) & \text{in Case 1} \\ [k-1]+(4k-1) & \text{in Case 2.}\end{cases}$$

If there were any doubt in the reader's mind about $G-v_{2m-1}$, then the reader could observe that φ restricted to $V\setminus\{v_{2m-1}\}$ is an (*L*,*k*)-coloring of that graph. Further φ can be modified to an (*L*,*k*)-coloring of $G-v_0$ by adding 1 to $\varphi(v_{2m-1})$, and to an (*L*,*k*)-coloring of $G-v_{2m-2}$ by adding $2k+2m-3$ in Case 1, and $4k-1$ in Case 2, to $\varphi(v_{2m-1})$. For $i\in[2m-3]$ we modify φ to an (*L*,*k*)-coloring φ^* on $G-v_i$ as follows:

In Case 1, if *i* is even, then for $0\le j<i$,

$$\varphi^*(v_j)=\begin{cases}(\varphi(v_j)\setminus\{i+1\})\cup\{i\} \text{ if } j \text{ is even} \\ (\varphi(v_j)\setminus\{i\})\cup\{i+1\} \text{ if } j \text{ is odd,}\end{cases}$$

for $i<j\le 2m-2$, $\varphi^*(v_j)=\varphi(v_j)$, and $\varphi^*(v_{2m-1})=\varphi(v_{2m-1})\cup\{i+1\}$; if *i* is odd then for

$$0\le j<i,\ \varphi^*(v_j)=\begin{cases}(\varphi(v_j)\setminus\{i+1\})\cup\{i\} \text{ if } j \text{ is odd} \\ (\varphi(v_j)\setminus\{i\})\cup\{i+1\} \text{ if } j \text{ is even,}\end{cases}$$

for $i<j\le 2m-2$, $\varphi^*(v_j)=\varphi(v_j)$, and $\varphi^*(v_{2m-1})=\varphi(v_{2m-1})\cup\{i\}$.

In Case 2, for $i\in[2k-1]$, if *i* is even then for $0\le j<i$,

$$\varphi^*(v_j)=\begin{cases}(\varphi(v_j)\setminus\{i+1\})\cup\{i\} \text{ if } j \text{ is even} \\ (\varphi(v_j)\setminus\{i\})\cup\{i+1\} \text{ if } j \text{ is odd,}\end{cases}$$

for $i<j\le 2m-2$, $\varphi^*(v_j)=\varphi(v_j)$, and $\varphi^*(v_{2m-1})=\varphi(v_{2m-1})\cup\{i+1\}$; if *i* is odd then for

$$0\le j<i,\ \varphi^*(v_j)=\begin{cases}(\varphi(v_j)\setminus\{i+1\})\cup\{i\} \text{ if } j \text{ is odd} \\ (\varphi(v_j)\setminus\{i\})\cup\{i+1\} \text{ if } j \text{ is even,}\end{cases}$$

for $i<j\le 2m-2$, $\varphi^*(v_j)=\varphi(v_j)$, and $\varphi^*(v_{2m-1})=\varphi(v_{2m-1})\cup\{i\}$. If $2k\le i\le 2m-2$, for $0\le j<i$, set $\varphi^*(v_j)=\varphi(v_j)$, and for $i<j\le 2m-2$,

$$\varphi^*(v_j)=\begin{cases}(\varphi(v_j)\setminus\{4k-2\})\cup\{4k-1\} \text{ if } j \text{ is odd} \\ (\varphi(v_j)\setminus\{4k-1\})\cup\{4k-2\} \text{ if } j \text{ is even,}\end{cases}$$

and $\varphi^*(v_{2m-1})=\varphi(v_{2m-1})\cup\{4k-1\}$.

The proof will be complete when we verify that (*) holds for G, that is

$$\sum_{x\in C}\alpha(G(x,L)) \ge nk = 2mk .$$

Case 1. For $1\le x\le 2m-3$, $G(x,L)$ is a path of order $x+1$ if x is odd and x if x is even, so $\alpha(G(x,L)) = \lceil x/2\rceil$. For $2m-2\le x\le 2k$, $G(x,L) = G - v_{2m-1}$, so $\alpha(G(x,L)) = m$. For $2k+1\le x\le 2m+2k-3$, $G(x,L)$ is a path of order $2m+2k-x-2$ if x is even, and of order $2m+2k-x-1$ if x is odd, so $\alpha(G(x,L)) = m+k-1-\left\lfloor \frac{x}{2}\right\rfloor$. Finally, for $x\in[k-1]+(2k+2m-3)$, $\alpha(G(x,L)) = 1$. Therefore,

$$\sum_{x\in C}\alpha(G(x,L)) = \sum_{x=1}^{2m-3}\lceil x/2\rceil + \sum_{x=2m-2}^{2k} m + \sum_{x=2k+1}^{2m+2k-3}(m+k-1-\lfloor x/2\rfloor) + (k-1)$$
$$= (m-1)^2 + m(2k-2m+3) + (m+k-1)(2m-3) - ((m+k-1)(m+k-2)-k^2) + k - 1$$
$$= 2mk + k - m + 1 \ge 2mk = kn.$$

Case 2. For $1\le x\le 2k-1$, $G(x,L)$ is a path of order $x+1$ if x is odd and x if x is even; thus $\alpha(G(x,L)) = \lceil x/2\rceil$. $G(2k,L) = G - v_{2m-1}$, so $\alpha(G(2k,L)) = m$. For $x = 2k+r$, $1\le r\le 2k-2$, $G(x,L)$ is a path of order $2m-2-r$ if r (and thus x) is even, and of order $2m$-1-r if r (and thus x) is odd; $\alpha(G(x,L)) = m-1-\left\lfloor \frac{r}{2}\right\rfloor$. Clearly $\alpha(G(4k-1,L)) = m-k$ and $\alpha(G(x,L)) = 1$ for $x\in[k-1]+(4k-1)$. Therefore,

$$\sum_{x\in C}\alpha(G(x,L)) = \sum_{x=1}^{2k-1}\lceil x/2\rceil + m + \sum_{r=1}^{2k-2}(m-1-\lfloor r/2\rfloor) + (m-k) + (k-1)$$
$$= k^2 + m + (m-1)(2k-2) - (k-1)^2 + (m-1)$$
$$= 2mk = kn.$$

□

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3. References

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